

# Multiple Representations Help Teachers and Students Understand a Geometry Problem

Erol Uzan, Indiana University  
Shelly Sheats Harkness, University of Cincinnati

*This narrative account begins in a high school classroom as we describe how students were mostly unengaged with a “Problem of the Week.” As observers in this setting, we sat in the back of the classroom and attempted to solve the problem: Choose any three vertices of a cube at random. What is the probability that any three vertices will form a right triangle? Because of our different answers to the problem and the struggles we experienced as we attempted to visualize a cube with triangles on the faces and in the interior space we later created concrete and virtual manipulatives. Additionally, we posed this problem in a mathematics methods course with preservice high school teachers and then discussed the use of enactive (concrete), iconic (pictorial), and symbolic representations (Bruner, 1966). The significance of using concrete manipulatives for some mathematics problems cannot be overstated.*

## Introduction

The significance of using concrete manipulatives for some mathematics problems cannot be overstated. Using concrete materials is helpful for understanding abstract concepts (McNeil & Jarvin, 2007; Vinson, 2001). Hatfield (1994) found students’ use of manipulatives decreased as the grade level increased from kindergarten to sixth grade. Although the National Council of Teachers of Mathematics (NCTM) recommends their use at all grade levels, manipulatives are frequently used by students in grades K-8 but not in high school courses (Hartshorn & Boren, 1990).

We were visiting a high school classroom and geometry students were supposed to be working individually on a “Weekly Journal Problem,” written on the whiteboard. Each week the students were required to solve a problem in their journals and they were given 5-10 minutes at the beginning of class to work on the problem. However on this day, most chit-chatted with each other, sat quietly, or raised their hands for help from the teacher.

*Each week the students were required to solve a problem in their journals and they were given 5-10 minutes at the beginning of class to work on the problem.*

They were not using concrete materials or manipulatives. Before you read the remainder of this article, we encourage you to solve the “Weekly Journal Problem” that the students were tackling.

**Problem: Choose any three vertices of a cube at random. What is the probability that any three vertices will form a right triangle?**

How did you approach the problem? What strategies did you use? How confident are you with your answer?

## Our Solution Attempts

We posed these questions because when we first attempted to solve the problem we had different solutions. Erol pictured cubes and triangles and used his knowledge of permutations to find the probability. Shelly drew a picture of a cube, labeled the vertices, attempted to imagine triangles on the faces and within the cube, and then created patterns. When the two of us sat in the back of the classroom and quietly compared our different answers we were not confident that

we had accounted for all possible triangles in the cube and, additionally, we were uncertain as to which triangles inside the cube were right.

If someone had videotaped us as we discussed the problem, the images would have shown the two of us first moving our arms and pencils to form right triangles (outlining the faces and edges of a cube) and then utilizing a tissue box from the teacher's desk as a concrete representation of a prism. We pointed to the lines and vertices formed by the walls and ceiling of the classroom, "drawing" imaginary lines on the surfaces and spaces. Employing these methods, we were able to visualize some triangles as two-dimensional shapes on and within a three-dimensional space.

We moved from these initial imaginings and attempted to use symbolic representations, the "Binomial Coefficient," to find all of the different ways to arrange three vertices out of the eight vertices on a cube. Next, we needed to find which of the 56 ( $8!/[3!(8-3)!]$ ) possible triangles were right. These maneuvers helped, but we left the high school classroom and continued to think about the "Weekly Journal Problem" after our initial deliberations.

In fact, it was only after we built manipulatives or models (see Figure 1) and reflected on the problem that we were convinced of a correct solution (see Figures 2a and 2b).

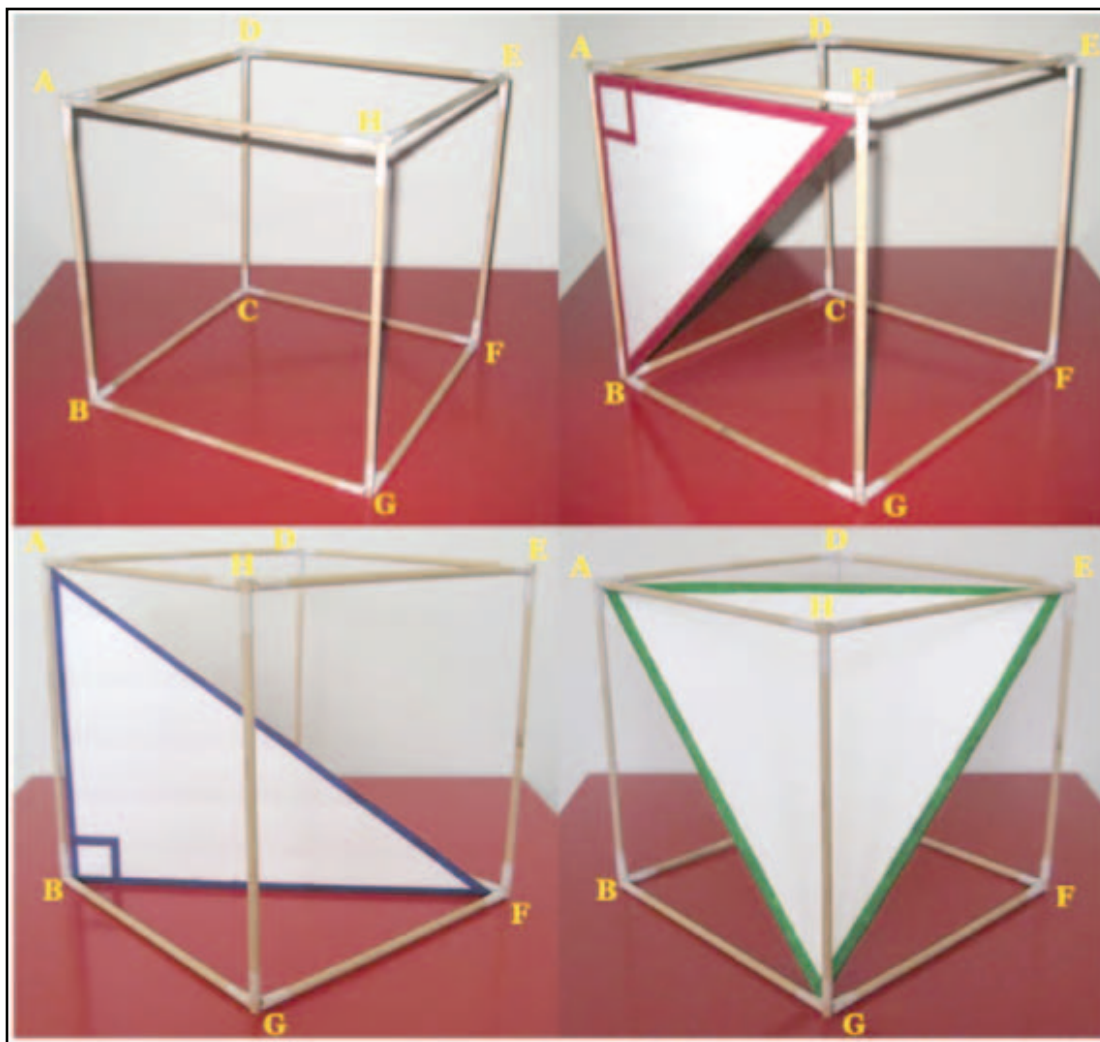


Fig 1 Concrete manipulatives

*We pointed to the lines and vertices formed by the walls and ceiling of the classroom, "drawing" imaginary lines on the surfaces and spaces.*

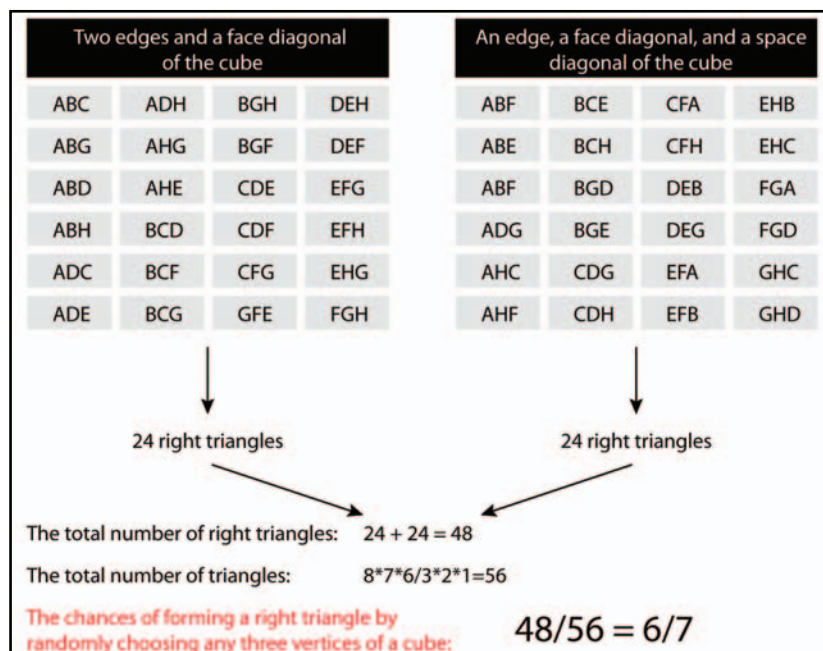


Fig 2a Solution to weekly journal problem

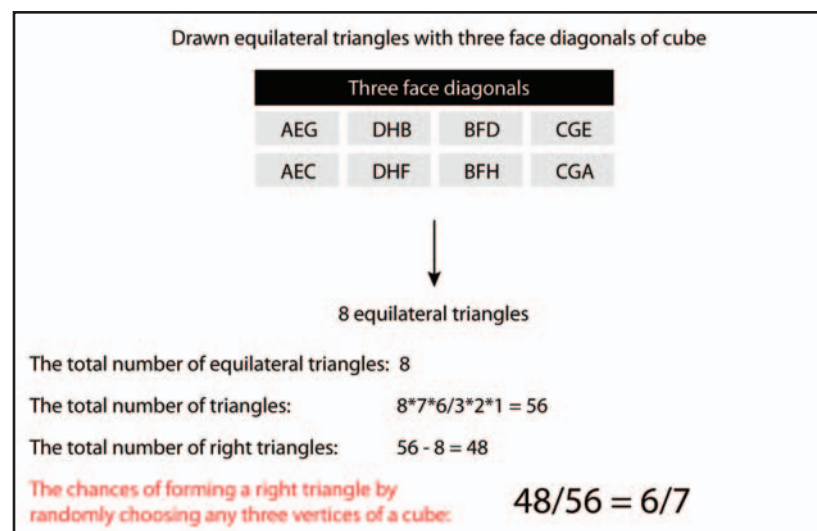


Fig 2b Solution to weekly journal problem

Furthermore, as mathematics educators we pondered pedagogical implications based on our struggles. We thought about the students in the high school classroom and considered the following questions: Were their struggles the same as ours? What representations were they using? How might concrete manipulatives have helped them make sense of the problem?

Hence, we reemphasize our initial sentence in this article: The significance of using concrete manipulatives for some mathematics problems cannot be overstated. We needed tools that we could touch and move; these supported our efforts to visualize the problem. From those visualizations we were able to create patterns in a symbolic representation (see Figure 2). Indeed, Bruner (1966) suggested that when we attempt to learn new mathematical ideas, it is valuable to follow a sequence through three modes of representation: enactive (action-based or concrete) to iconic (image-based or pictorial) to symbolic (language-based or letters and numerals). This is true for adult learners as well as for young children (Bruner, 1966).

The significance of using concrete manipulatives for some mathematics problems cannot be overstated. We needed tools that we could touch and move; these supported our efforts to visualize the problem.

## Using the Problem in a Preservice Teachers' Methods Course

Because of our fascination with this problem, we decided to pose it in a methods course for preservice high school teachers in order to investigate how they attempted to solve it. We also wanted them to reflect upon the implications of allowing their future students to use or not use manipulatives for solving problems. Please note these students were seniors who were earning dual majors in mathematics and education. Rather than working individually (as was the case in the high school classroom which we observed), the preservice teachers worked in groups of three. Similar to the enactment in the high school classroom, we gave the preservice teachers no manipulatives and posed the problem. We supplied them with chart paper and markers and walked around the classroom in order to listen to their strategies. All groups of preservice teachers drew cubes on their chart paper but none of them labeled the vertices in order to keep track of the triangles in an organized way. They seemed to move directly from their sketches of cubes to using symbolic representations with numerals, factorials, and ratios.

After about 15 minutes of allowing the groups to explore the problem, we brought the class together as a whole group and asked them to share their thinking about the problem. Needless to say, during this conversation they told us they “wanted” and “needed” concrete manipulatives – cubes and triangles – in order to be more confident about their strategies. Following this conversation, we gave the groups cubes (similar to the ones shown in Figure 1), scissors, and foam board. For a second time, they explored the problem in their groups.

Using the foam board, all groups created right triangles with legs that were the same dimensions as the edges of the cubes and with hypotenuses that were the same dimensions

as the face diagonals of the cube. One group of preservice teachers used the Pythagorean Theorem to calculate whether or not the triangles formed within the cube were right; they measured three edge lengths of the triangles they created and checked those with the theorem.

At the end of the exploration, all groups shared their findings and solution methods with the entire class. They correctly found the number of possible triangles (56) by using the concrete cube and the number of right triangles formed on the faces of the cube (24). However, they still had difficulty determining which triangles inside the cube were right (24) and which were equilateral (8). None of the groups found the correct answer (48/56 or 6/7).

With this lesson, the preservice teachers gained experience with manipulatives and they worked with concrete objects to see the triangles in different perspectives. At the end of the lesson, they shared their thoughts about using manipulatives and they indicated that even though their solutions were incorrect, the manipulatives helped them to explore the problem. This led to a conversation about teaching and learning within the context of Bruner's (1966) three modes of representation.

## Extending the Lesson with Technology

The authors of NCTM's Principles and Standards for School Mathematics (2000) note, “Technology is an essential tool for teaching and learning mathematics effectively; it extends the mathematics that can be taught and enhances students' learning” (p. 24). Thus, we decided to extend our own exploration with the “Weekly Journal Problem” and Erol created a virtual manipulative with GeoGebra (see Figure 3), an open source dynamic mathematics software for learning and teaching geometry, algebra, and calculus concepts. The sketch

*With this lesson, the preservice teachers gained experience with manipulatives and they worked with concrete objects to see the triangles in different perspectives.*

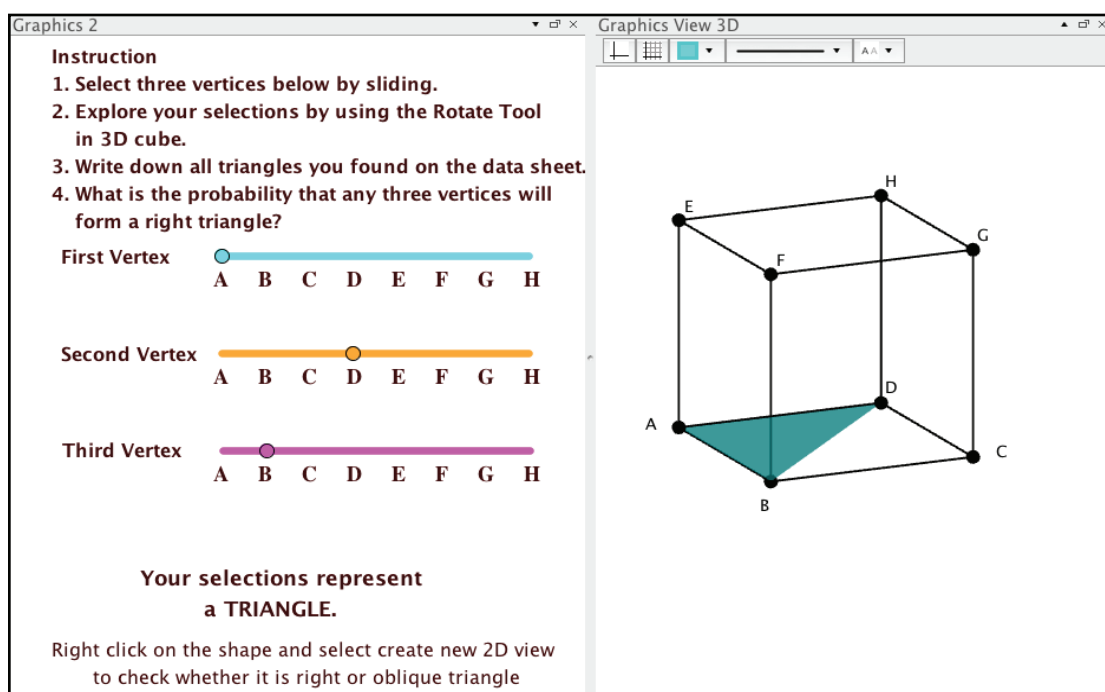


and directions on its use is available from the MathWorld (<http://the-math-world.blogspot.com/2011/04/geogebra-5.html>). A portion of the sketch is illustrated in Figure 3. Students use the sketch as they write down triangles they find on the datasheet provided at the end of this article.

To use the GeoGebra applet, students complete the following steps.

1. Download GeoGebra 5.0 3D Beta Version (<http://www.geogebra.org/forum/viewtopic.php?f=52&t=19846>) for Virtual Manipulatives
2. Save Right Triangle/Cube Virtual Manipulatives & Activity Sheet (<http://www.geogebra.org/en/upload/index.php?&direction=0&order=&directory=erol>) to your computer
3. Launch GeoGebra 5.0
4. Click File/Open/Select the saved file
5. Manipulate sliders (in left pane) and rotate tool (in right pane) to explore possible solutions

*The virtual manipulative includes instructions and sliders so that students can select three vertices on the cube. They can rotate, turn, and zoom-in and zoom-out of the cube with mouse clicks.*



**Fig 3** GeoGebra 5.0 sketch of weekly journal problem (Available on-line at <http://the-math-world.blogspot.com/2011/04/geogebra-5.html>)

Recall that when the preservice teachers were working on the problem with the manipulatives, they did not use letters or numbers to name the vertices of the cube. The lack of vertex labeling may have prevented them from seeing the patterns in the symbolic representation and caused them to make mistakes when they were calculating the number of right and non-right (equilateral) triangles. Therefore, we labeled the vertices of the dynamic cube in the virtual manipulative with letters.

The virtual manipulative includes instructions and sliders so that students can select three vertices on the cube. They can rotate, turn, and zoom-in and zoom-out of the cube with mouse clicks. Additionally, the virtual manipulative allows students to recognize that one triangle can be named in six different ways. To be specific, when three vertices are selected in different orders, all selected vertices represent the same triangle and this triangle is shown with the same color in the virtual manipulative (see Figures 4 and 5).

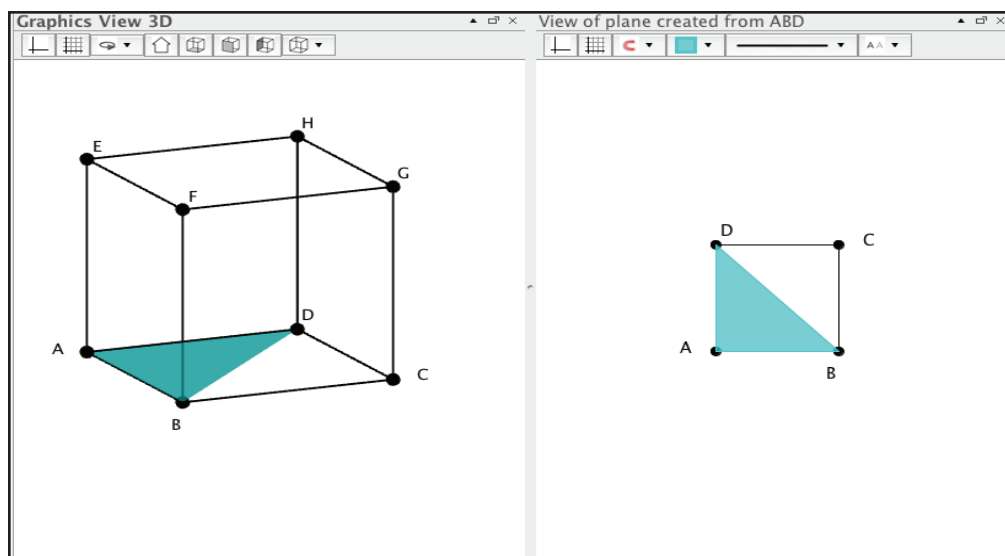


Fig 4 Using plane view in GeoGebra 5.0 to informally verify that triangle ABD is right

<b>First vertex A</b> ABC ACB ADB AEB AFB AGB AHB ABD ACD ADC AEC AFC AGC AHD ABE ACE ADE AED AFD AGD ACB ABF ACF ADF AEF AFE AGE AHE ABG ACG ADG AEG AFG AGF AHF ABH ACH ADH AEH AFH AGH AHG							
<b>First vertex B</b> BAC BCA BDA BEA BFA BGA BHA BAD BCD BDC BEC BFC BGC BHC BAE BCE BDE BED BFD BGD BHD BAF BCF BDF BEF BFE BGE BHE BAG BCG BDG BEG BFG BGF BHF BAH BCH BDH BEH BFH BGH BHG							
<b>First vertex C</b> CAB CBA CDA CEA CFA CGA CHA CAD CBD CDB CEB CFB CGB CHB CAE CBE CDE CED CFD CGD CHD CAF CBF CDF CEF CFE CGE CHE CAG CBG CDG CEG CFG CGF CHF CAH CBH CDH CEH CFH CGH CHG							
<b>First vertex D</b> DAB DBA DCA DEA DFA DGA DHA DAC DBC DCB DEB DFB DGB DHB DAE DBE DCE DEC DFC DGC DHC DAF DBF DCF DEF DFE DGE DHE DAG DBG DCG DEG DFG DGF DHF DAH DBH DCH DEH DFH DGH DHG							
<b>First vertex E</b> EAB EBA ECA EDA EFA EGA EHA EAC EBC ECB EDB EFB EGB EHB EAD EBD ECD EDC EFC EGC EHC EAF EBF ECF EDF EFD EGD EHD EAG EBG ECG EDG EFG EGF EHF EAH EBH ECH EDH EFH EGH FHG							
<b>First vertex F</b> FAB FBA FCA FDA FEA FGA FHA FAC FBC FCB FDB FEB FGB FHB FAD FBD FCD FDC FEC FGC FHC FAE FBE FCE FDE FED FGD FHD FAG FBG FCG FDG FEG FGE FHE FAH FBH FCH FDH FEH FGH FHG							
<b>First vertex G</b> GAB GBA GCA GDA GEA GFA GHA GAC GBC GCB GDB GEB GFB GHB GAD GBD GCD GDC GEC GFC GHC GAE GBE GCE GDE GED GFD GHD GAF GBF GCF GDF GEF GFE GHE GAH GBH GCH GDH GEH GFH GHF							
<b>First vertex H</b> HAB HBA HCA HDA HEA HFA HGA HAC HBC HCB HDB HEB HFB HGB HAD HBD HCD HDC HEC HFC HGC HAE HBE HCE HDE HED HFD HGD HAF HBF HCF HDF HEF HFE HGE HAG HBG HCG HDG HEG HFG HGF							

Fig 5 Patterns of color-coding

Another advantage that the virtual manipulative offers is that students have a chance not only to see the selected triangle on the three-dimensional cube but also investigate the same triangle in the two-dimensional view by right clicking on the cube. Additionally, students can explore whether or not the selected triangle is right by measuring its edges (and then using the Pythagorean Theorem). Although we did not use this virtual manipulative with the preservice teachers, we feel as though it could provide students with the capability to use iconic (image-based or pictorial) representations to solve the problem. Additionally, the color-coding in Figures 4 and 5 could lead to interesting discussions about combinations and permutations.

Another advantage that the virtual manipulative offers is that students have a chance not only to see the selected triangle on the three-dimensional cube but also investigate the same triangle in the two-dimensional view

## Conclusion: Concrete, Pictorial, and Symbolic Representations

When students are learning new content with concrete manipulatives, they will experience mathematics in an enactive mode. They can move to another mode of representation, iconic, by drawing pictures or using virtual manipulatives. These concrete and iconic representations can ameliorate students' use of symbolic representations via patterns, symbols, formulas, and tables. As a result, we believe that experiences with both physical and virtual manipulatives can impact teaching and learning mathematics because students who need these supports (much like we did as we sat in the high school classroom and worked on the "Problem of the Week") can move through Burner's (1966) modes of representation to the symbolic or abstract. Therefore, the importance of allowing high school students opportunities to use concrete manipulatives to solve some mathematics problems cannot be overstated.

## References

- Bruner, J. S. (1966). *Toward a theory of instruction*. Cambridge, MA: Belkapp Press.
- Hartshorn, R., & Boren, S. (1990). *Experiential learning of mathematics: Using manipulatives*. ERIC Digest. Charleston, WV: ERIC Clearinghouse on Rural Education and Small Schools. (ERIC Document Reproduction Service No. ED321967)
- Hatfield, M. M. (1994). Use of manipulative devices: Elementary school cooperating teachers' self report. *School Science and Mathematics*, 94(6), 303-309.
- McNeil, N. M., & Jarvin, L. (2007). When theories don't add up: Disentangling the manipulatives debate. *Theory Into Practice*, 46(4), 309-316.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Vinson, B. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal*, 29(2), 89-95.



**EROL UZAN**, [eroluzan@indiana.edu](mailto:eroluzan@indiana.edu), is a doctoral student in mathematics education at Indiana University, Bloomington. He is interested in the role of technology on mathematics teaching and learning, creation and investigation of technology-intensive mathematics curricula, and mathematics teacher preparation.



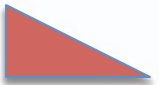

**SHELLY SHEATS HARKNESS**, [shelly.harkness@uc.edu](mailto:shelly.harkness@uc.edu), is an Associate Professor at the University of Cincinnati. Her research interests include Ethnomathematics, mathematics and social justice, and teachers' use of Elbow's "believing game" as they attempt to understand students' wrong answers to problems.

# Right Triangle/Cube

NAME \_\_\_\_\_

**Problem:** Choose any three corners of a cube at random. What are the chances that those points will form a right triangle?

1. Write down your finding under the appropriate category.

<i>Right Triangles</i> 	<i>Oblique Triangles</i> 

2. Number of triangles : \_\_\_\_\_

3. Number of right triangles : \_\_\_\_\_

4. The probability of the event : \_\_\_\_\_